

Measurement-induced quantum trajectories, Brownian flows, and weak stochastic ratchets

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We consider a qubit governed by a sequence of weak measurements, with the measurement strength modified in a time- and state-dependent manner. The resulting trajectory of the qubit in the phase space has many in common with a trajectory of a Brownian particle in an time- and space-dependent potential. In particular, we show a possibility of a weak form of a stochastic ratchet phenomenon. Furthermore, if the weak measurement strength approaches small value or even zero in a way conditioned to some particular state, a dynamical localization of the “particle” takes place. A singularity can be created in this way, which behaves like an “artificial basis state”.

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I. INTRODUCTION

Dynamics of quantum systems with the states frequently monitored by a measurement apparatus can be rather controversial and is a subject of constant interest over the years [1–15]. Every measurement in the measurement sequence may be tuned to cause only a partial collapse of the system’s wavefunction, with the collapse effect being arbitrary weak (so called “weak measurements”) [4, 8, 16–18]. In another context, the term “weak measurement” was introduced by Aharonov, Albert, and Vaidman (AAV) [19] as being attributed to a measurement of a continuous degree of freedom (e. g. an electron position) coupled to a discrete one (spin) via post-selection. These two approaches were recently shown to be equivalent [20, 21]. As a result of repetitive application of weak measurements — the situation which is sometimes referred to as weak Zeno measurements (WZM) — a kind of stochastic “quantum trajectory” arises [3, 14, 22–24] due to unpredictable character of every particular measurement outcome.

Just repeating the weak measurements, without any further action on the system, allows to control the system state in various ways. For instance, one can achieve an arbitrary state from any other one by repeating weak measurements in different bases [2, 5, 6, 12, 13, 25, 26]. Alternatively, by allowing the strength of the weak measurements (in AAV sense) to depend on the spatial coordinate, one can create a “potential wall” which may reflect a particle [12, 13].

In contrast, if we repeat a weak measurement of a single qubit in a fixed basis, the resulting dynamics was up to now believed to be very trivial. The resulting quantum trajectory just stochastically approaches one of the two qubit’s basis states $|0\rangle$ or $|1\rangle$. In this article we add a new dimension to this seemingly trivial dynamics by allowing the measurement strength to be changed in time and in a state-conditional way.

We observe that the stochastic equations, describing quantum trajectories in such simple one-dimensional system are in fact quite similar to the ones describing motion

of a small Brownian particle in a fluid flow (overdamped Brownian motion) [23, 27, 28]. One of the striking phenomena in such flows as well as in many other stochastic systems are so called stochastic ratchets. Namely, by varying the potential acting on the Brownian particle in time and/or space in a periodic way, it is possible to create an effective additional force, despite the potential introduces no average force [27–29]. Such ratchets are encountered in many stochastic systems of different nature [27, 28, 30–33]. Brownian ratchets are deeply connected to so called Parrondo games, when two or more lossy games are combined to give a winning one [30, 34, 35]. Very recently, the notion of weak Parrondo games and weak Brownian ratchets were introduced in [30] to describe the situation when two or more lossy games are combined to give just less lossy (but not winning) one. We remark that, although both Parrondo games and Brownian ratchets were considered in context of quantum systems [28, 36–39], this was up to now done via some direct action on the system itself.

In contrast, in the present article the situation of control-without-direct-action will be discussed. Here we show how the effective potential arising in WZM dynamics can be modified by changing the strength of the measurement periodically in time and in a state-conditioned fashion (that is, in dependence on the current system state). We demonstrate that the stochastic ratchet effect, albeit weak, is possible in such situation. We also demonstrate another interesting effect, namely dynamic localization of the state in the case when the measurement strength nearly vanishes at some particular system state. As a limiting case for this behaviour, a “false basis state” may appear, which “attracts” the stochastic trajectories in similar way as the true basis states do.

The article is organized as follows: in Sec. II we introduce the system under consideration and derive the master equation governing the probability distribution on the line between two basis states; In Sec. III we derive the equation for the continuous case taking into account conditionally-dependent measurement strength; In Secs. IV–VI we investigate the effects related to the con-

ditionally modified measurements strength. Finally, the conclusions and discussion are presented in Sec. VII.

II. DISCRETE DYNAMICS

A. The setting

Our model of weak Zeno measurements is depicted in Fig. 1(a) and uses projective measurements of ancilla qubit $|a\rangle$ to realize the weak ones of $|q\rangle$. The system is prepared in the state $|q\rangle \otimes |0\rangle$ with $|q\rangle = \cos\theta |0\rangle + \sin\theta |1\rangle$ for some θ . First, we apply a rotation $R_y(2\delta)$ to the ancilla state; the rotation R_y is conditioned to $|q\rangle = |1\rangle$ and is defined as:

$$R_y(\delta) |0\rangle \mapsto \cos\delta |0\rangle + \sin\delta |1\rangle, \quad (1)$$

$$R_y(\delta) |1\rangle \mapsto \cos\delta |1\rangle - \sin\delta |0\rangle. \quad (2)$$

Afterwards, we unconditionally apply the rotation R_y by some other angle α and finally we measure the ancilla qubit. If $\alpha \neq 0$ and δ is small, the resulting measurement modifies $|q\rangle$ only slightly. After the measurement, we repeat the whole procedure using another ancilla in the initial state $|0\rangle$ (or the same ancilla returned to the state $|0\rangle$).

The state of the whole system $|q\rangle \otimes |0\rangle$ [see Fig. 1(a)] is transformed by two R_y operators described above as:

$$|q\rangle \otimes |0\rangle \mapsto |q_0\rangle \otimes |0\rangle + |q_1\rangle \otimes |1\rangle, \quad (3)$$

$$|q_i\rangle = \sum_j b_{ij} |j\rangle, \quad (4)$$

where b_{ij} are $(i+1, j+1)$ th element of the matrix b defined as:

$$b = \begin{pmatrix} \cos\theta \cos\alpha & \sin\theta \cos(\delta + \alpha) \\ \cos\theta \sin\alpha & \sin\theta \sin(\delta + \alpha) \end{pmatrix}. \quad (5)$$

If the measurement of $|a\rangle$ gives 0, the system state is reduced to $|q\rangle = |q_0\rangle / \sqrt{p_0}$ and in the opposite case to $|q\rangle = |q_1\rangle / \sqrt{p_1}$, where

$$p_0 = \cos^2\alpha \cos^2\theta + \sin^2\theta \cos^2(\delta + \alpha), \quad (6)$$

$$p_1 = \cos^2\theta \sin^2\alpha + \sin^2\theta \sin^2(\delta + \alpha). \quad (7)$$

The probabilities of the corresponding outcomes are p_0 and p_1 .

The above description can be reformulated in the form of a generalized measurement formalism, with the measurement operators

$$\mathcal{B}_0 = b_{11} |0\rangle\langle 0| + b_{12} |1\rangle\langle 1|, \quad (8)$$

$$\mathcal{B}_1 = b_{21} |0\rangle\langle 0| + b_{22} |1\rangle\langle 1|, \quad (9)$$

so that $\sum_j \mathcal{B}_j \mathcal{B}_j^\dagger = 1$ and the state after the measurement with the result j is transformed as: $|q\rangle \rightarrow \mathcal{B}_j |q\rangle / \sqrt{p_j}$.

The resulting process is a (classical) one-dimensional random walk along the axis θ as shown in Fig. 1(b). It

spends most of time in the vicinity of the limiting states $|q\rangle = |0\rangle$ and $|q\rangle = |1\rangle$ ($\theta = 0, \pi/2$). It is thus useful to introduce parabolic coordinates [17] as:

$$x = \operatorname{atanh}\{-\cos(2\theta)\}; \quad \theta = \arcsin \sqrt{\frac{1 + \tanh x}{2}}. \quad (10)$$

In this coordinate system, $\theta = 0$ corresponds to $x = -\infty$ and $\theta = \pi/2$ corresponds to $x = +\infty$ [(cf. Fig. 1(b)). Using x instead of θ allows to expand these vicinities into semi-infinite intervals. We also note that if we measure $|q\rangle$ directly (instead of $|a\rangle$), the probability to find $|q\rangle$ in the state $|1\rangle$ will be:

$$\Pi = \sin^2\theta = (1 + \tanh x)/2. \quad (11)$$

B. Classical random walk interpretation

Now, for the sake of simplicity, we exclude the situation when $|q\rangle$ is exactly in one of the basis states $|0\rangle$ or $|1\rangle$ in the beginning of the process. In this case, the equations above allow to define the process as a one-dimensional random walk on the line $x \in (-\infty, +\infty)$ in the following way: assuming that at the n th iteration step the system is in the point x_n , at the $n+1$ step it will be in the point either $x_{n+1} = x_n + \epsilon_0$ or $x_{n+1} = x_n + \epsilon_1$ (depending on the measurement outcome), every of two variants occurring with the probabilities $p_i(x_n)$, $i = 0, 1$. Few realizations of this random walks are shown in Fig. 1(b).

The probabilities $p_i(x)$ can be rewritten in x -coordinates as:

$$p_0(x) = \Pi(x) \cos^2(\delta + \alpha) + (1 - \Pi(x)) \cos^2\alpha. \quad (12)$$

$$p_1(x) = \Pi(x) \sin^2(\delta + \alpha) + (1 - \Pi(x)) \sin^2\alpha. \quad (13)$$

The step sizes ϵ_i , $i = 0, 1$, do not depend on x and are given by (see details in A):

$$\epsilon_0 = \operatorname{atanh}\left(\frac{2 \cos^2(\alpha + \delta)}{\cos^2(\delta + \alpha) + \cos^2\alpha} - 1\right), \quad (14)$$

$$\epsilon_1 = \operatorname{atanh}\left(\frac{2 \sin^2(\alpha + \delta)}{\sin^2(\delta + \alpha) + \sin^2\alpha} - 1\right). \quad (15)$$

Eqs. (12)-(15) define obviously a Markovian random walk.

C. Conditionally varied measurement parameters

Suppose, we know exactly the initial state of the system x_0 and are able to make all the rotations also exactly. In this case, the subsequent positions x_n of the qubit on x -line can be also calculated exactly since we know the measurement outcomes $M_n = 0, 1$ and thus the step sizes ϵ_i at every n . We may now introduce the state-dependent dynamics by allowing the parameters δ, α to be dependent on the step number n and the state of the system at

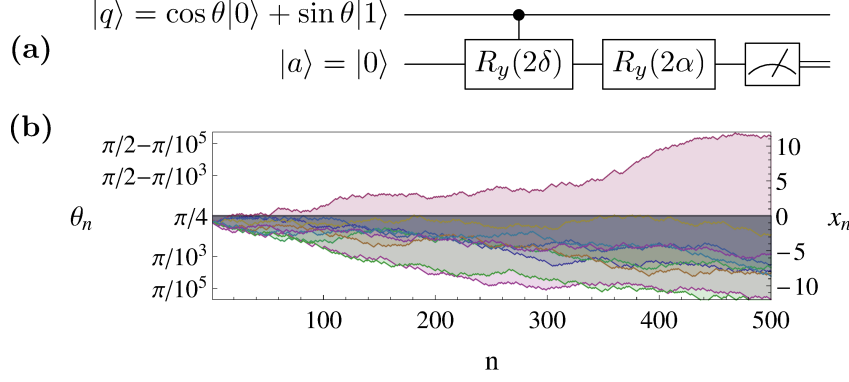


FIG. 1. (a) The model of weak measurements of the qubit $|q\rangle$ using an ancilla $|a\rangle$ (initially in the state $|0\rangle$) and its rotations R_y , followed by the measurement of $|a\rangle$. After the measurement, the process is repeated with the same or another ancilla in the state $|0\rangle$. In (b), the stochastic process induced by repeating application of (a) using θ coordinates (left y-axis) and x -coordinates given by Eq. (10) (right y-axis) vs. measurement number n is shown. The x -coordinates are obviously better suitable to study the asymptotic behaviour as $n \rightarrow \infty$.

the last step x_n . That is we may take some (pre-defined) functions of two arguments $\alpha(n, x)$, $\delta(n, x)$ and at every step select the parameters for the next step α_{n+1} , δ_{n+1} as: $\alpha_{n+1} = \alpha(n, x_n)$, $\delta_{n+1} = \delta(n, x_n)$. In this way, the parameters p_i , ϵ_i of our random walk are also some pre-defined functions of n , x_n defined by Eqs. (12)-(15).

The functions $\alpha(m, x)$, $\delta(m, x)$ may be quite arbitrary. They add new degrees of freedom to our system, leading, as we will see, to rather interesting new dynamics. We remark also that in quantum control schemes [14] the information about the current state of the system is often used by feeding it back into the system via modification of the system's Hamiltonian. In contrast, in our case, only the parameters of the measurement itself, but not the parameters of system, are changed.

D. Master equation

Using Eqs. (3)-(5) or Eqs. (8)-(9) it is easy to obtain an equation governing the evolution of the probability density function (pdf) $P(n, x)$, describing the probability P of $|q\rangle$ to appear in the vicinity of x at the step n . Since our qubit $|q\rangle$ always remains in a pure state which is fully described by its coordinate x (or, equivalently, by θ), such master equation is just another way to express the dynamics of $|q\rangle$. It provides essentially the same information as Eqs. (3)-(5) or Eqs. (8)-(9). This reformulation will be however useful in the next sections when we consider stochastic ratchet behaviour.

We start from the general case with no assumption about the particular coordinate system. We use the variable y by which we may understand any of the coordinates x , θ or Π mentioned before. We introduce furthermore the measure $dP(n, y) = P(n, y)dy$ which expresses simply the total probability to find $|q\rangle$ in the interval $[y, y + dy]$. Then, by definition of our process, using the

Markov property and the formula for total probability [40] we obtain the following relation:

$$dP(n+1, y) = p_0(y_0(y))dP(n, y_0(y)) + p_1(y_1(y))dP(n, y_1(y)), \quad (16)$$

where $y_i(y)$, $i = 1, 2$ are defined in an implicit way as $y = y_i + \epsilon_i(y_i)$. This expression is valid for an arbitrary (also varying) step size, that is, also for the state-conditioned trajectories as they were defined above in the previous section. In the case of x -coordinates ($y \equiv x$) we obtain straightforwardly the following expression for $P(n, x)$:

$$P(n+1, x) = p_0(x_0(x))x'_0(x)P(n, x_0(x)) + p_1(x_1(x))x'_1(x)P(n, x_1(x)), \quad (17)$$

where $x = x_i(x) + \epsilon_i(x_i(x))$, $x'_i(x) = dx_i(x)/dx$. In particular, for the constant measurement strength we have, $\epsilon_i = \text{const}$, $x'_i(x) = 0$, and therefore we have:

$$P(n+1, x) = p_0(x - \epsilon_0)P(n, x - \epsilon_0) + p_1(x - \epsilon_1)P(n, x - \epsilon_1). \quad (18)$$

Very important are conserved quantities of Eq. (16) or Eq. (17). The most obvious one is the average value of Π on n th step, $\langle \Pi \rangle_n \equiv \int_{-\infty}^{+\infty} \Pi(x)P(n, x)dx$, which represents the a priori probability to find $|q\rangle$ in the state $|1\rangle$ if we perform a projective measurement of $|q\rangle$ after the n -th step of our process. One can show that from Eq. (18) it follows that:

$$\langle \Pi \rangle_{n+1} = \langle \Pi \rangle_n, \quad (19)$$

and thus for any n , $\langle \Pi \rangle_n = \langle \Pi \rangle_0$. Eq. (19) can be obtained by substituting Eq. (17) into definition of $\langle \Pi \rangle_{n+1}$,

giving thus

$$\langle \Pi \rangle_{n+1} = \int_{-\infty}^{+\infty} \Pi(x) P(n+1, x) dx = \int_{-\infty}^{+\infty} P(n, x) \{p_0(x)\Pi(x) + p_1(x)\Pi(x)\} dx, \quad (20)$$

where we made a replacement $x'_i(x)dx \rightarrow dx_i$ and the variable change $x_i(x) \rightarrow x$ in both parts of the integral. Since $p_0(x) + p_1(x) = 1$, this gives Eq. (19). We remark that Eq. (19) is universal, that is valid for any choice of the measurement parameters, also if they vary in dependence on the step n or current position x_n .

III. CONTINUOUS DIFFUSIVE LIMIT

A. general equation

The continuous limit arises if we tend the measurement strength to zero. In this case, instead of the discrete equation Eq. (17), a continuous equation arises, with the step numbers n being mapped to a continuous “time” t . If the measurement strength is constant (independent on n) and if this constant strength tends to zero, the corresponding limit is universal, that is, does not depend on the measurement strength and on the particular measurement procedure. The dynamics in such “unconditional” continuous limit is often described by the stochastic Schrödinger equation or by the master equation for the density matrix [14, 18, 22–24, 41]. Nevertheless, to our knowledge, a consideration general enough to include conditionally varied measurements were presented only very recently in [42]. Earlier works dealt only with the case of measurements of equal strength or at least time-independent ones. Instead of directly writing the resulting equation according [42] we will proceed, for the sake of closeness of presentation, from the master equation for $P(n, x)$, derived in the previous section, to the corresponding continuous limit described by the Fokker-Planck (FP) equation. The FP approach used here is also different from [42] where Ito calculus is used, but Ito and FP approaches are, of course, equivalent [40]. We use the later because of the straightforward connection to the methods used in the theory of stochastic ratchets [28].

That is, our goal here is to derive the FP equation in the case which includes the walk with conditionally varying measurement parameters δ , α which depend on the outcome of the all previous measurements and also on n . The transition to the continuous time can be done as follows: We introduce “time” t such that each step of our process corresponds to a small interval $\tau_n = \tau(\delta_n(n, x_{n-1}), \alpha_n(n, x_{n-1}))$, that is, we replace $n = \sum_{i=1}^n 1$ by

$$t \equiv \sum_{i=1}^n \tau_i \quad (21)$$

and allow $\tau_i(x_i)$ to tend to zero for every i , x_i . We do not assume that all τ_i are equal. In our case, as $\tau_i(x_i) \rightarrow 0$, we can expect that $P(t, x) \equiv P(n, x)|_{n \rightarrow t}$ changes at every step only slightly and we can then decompose $P(t, x)$ into series as:

$$P(t + \tau_n, x) \approx P(t, x) + \tau_n \partial_t P(t, x). \quad (22)$$

To be allowed to do this we must assume that, independently on n , the step size $\epsilon_{i,n} = \epsilon_i(\delta_n(n, x_{n-1}), \alpha_n(n, x_{n-1}))$ defined in Eqs. (14)-(15) goes to zero as $\tau_n \rightarrow 0$. In particular, this is the case if $\delta_n \rightarrow 0$, $\alpha_n = \text{const}_n > 0$ for all n . Thus, for small enough δ_n , we may assume:

$$\alpha_n = \text{const}(n, x), \quad (23)$$

$$\delta_n = \delta g_\delta(x_{n-1}, n), \quad (24)$$

$$\tau_n = \delta^2 g_\tau(x_{n-1}, n), \quad (25)$$

where we introduced the parameter $\delta \rightarrow 0$ which describes how fast δ_n and τ_n approach to zero; $g_\tau(x, n) > 0$, $g_\delta(x, n)$ are some functions which do not depend on δ and which we can chose at our will.

That is, we require that all τ_n , δ_n tend to zero as $O(\delta^2)$ and $O(\delta)$ respectively. This template is taken from the consideration of the case with the constant step size as shown in Appendix B. The functions $g_\tau(x, n) > 0$ and $g_\delta(x, n)$ provide “form-factors”, which determine the strength of measurement in dependence on the system position x and n . Using Eqs. (21),(24),(25), we define a function $g(x, t)$ as:

$$g(x, t) = \left. \frac{g_\delta(n, x_{n-1})}{g_\tau(x_{n-1}, n)} \right|_{n \rightarrow t; x_{n-1} \rightarrow x}. \quad (26)$$

Using Eqs. (22),(26) we derive, in a rather standard way, the FP equation (see Appendix C for details and a description of the general procedure in [40]):

$$\partial_t P(t, x) = -\partial_x J(t, x), \quad (27)$$

$$J(x, y) = \mu(t, x)P(t, x) - \partial_x (D(t, x)P(t, x)). \quad (28)$$

Here

$$\mu(x, t) = g(x, t)^2 \tanh(x), \quad D(x, t) = g(x, t)^2 / 2, \quad (29)$$

have now the meaning of the drift and diffusion coefficients, respectively.

This FP equation, as said, describes the dynamics of the pure state $|q\rangle$ which position on the line between $|0\rangle$ and $|1\rangle$ is described by the coordinate x . Stochastic distribution of the position x is due to unpredictable character of the weak measurement sequence. The same FP equation describes also a Brownian heavily damped particle moving in the potential

$$V(x, t) = - \int_0^x \mu(x', t) dx', \quad (30)$$

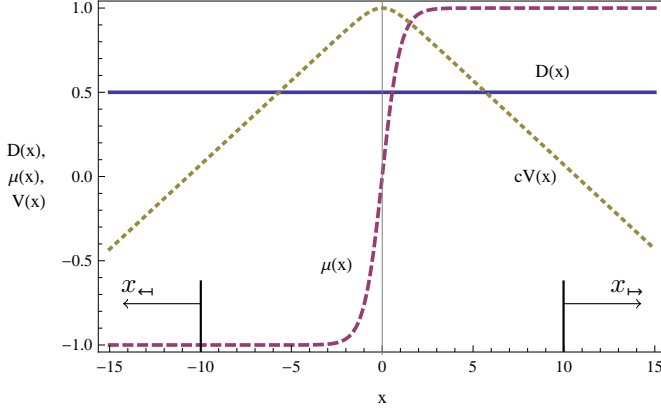


FIG. 2. Diffusion $D(x)$ (solid blue line), drift coefficient $\mu(x)$ (dashed red line) and the effective potential $V(x)$ (dotted yellow line) in dependence on x according to Eq. (32). The asymptotic coordinates $x_{\leftarrow}, x_{\rightarrow}$ defined in Eqs. (34)-(35) are shown, with $X = -10$ in this case. Asymptotic coordinates are useful when $|x|$ is large, that is, as the step $n \rightarrow \infty$: $D(x)$, $\mu(x)$ and $V(x)$ are significantly simplified for large $|x|$.

[28, 43]. The average drift velocity $\langle \dot{x} \rangle \equiv \int_{-\infty}^{+\infty} \frac{dx}{dt} P(x, t) dx$ can be obtained also as an average of $J(x, t)$:

$$\langle \dot{x} \rangle = \int_{-\infty}^{+\infty} J(x, t) dx. \quad (31)$$

For the case of $g(x) = 1$, that is, if the step size in our random walk is state-independent, we have:

$$\mu(x) = \tanh(x), \quad V(x) = \ln(\cos(x)), \quad D = \frac{1}{2}. \quad (32)$$

The corresponding functions μ , D , V are shown in Fig. 2. The solution of Eqs. (27)-(28), (32) with the initial condition $P(0, x) = \delta(x - X)$, where $\delta(x - X)$ is the Dirac delta-function localized at the arbitrary point X can be found analytically [23]:

$$P(t, x) = \frac{1}{\sqrt{2\pi t}} \frac{\cosh x}{\cosh X} \exp\left(-\frac{t^2 + (x - X)^2}{2t}\right). \quad (33)$$

B. Asymptotic FP equation

To make a semi-analytic approach described in [28] working (as described in the next section), we have to find the conditions where, assuming $g = \text{const}$ in Eq. (32), we have also $\mu = \text{const}$. In our equations this is generally not the case because of $\tanh(x)$ factor. However, as one can see from Fig. 2 and from Eq. (32), the deviation from this condition decreases exponentially with $|x|$ because

$|\tanh(x)|$ exponentially fast approaches 1. Also, as one can see from Eq. (33), if we take the initial starting point X far away from the origin $X = 0$, $P(t, x)$ behaves very much like a normal Gaussian distribution which shifts with time with the constant unit speed away from $x = X$, and expands with the variance $\sigma^2 = t$.

This allows us to consider the asymptotic behavior, as the initial point X and thus x are far enough from the origin $x = 0$. We thus introduce “shifted” coordinates $x_{\leftarrow}, x_{\rightarrow}$ as (see also Fig. 2):

$$x_{\leftarrow} = x + X, \quad (34)$$

$$x_{\rightarrow} = x - X, \quad (35)$$

where $X \gg 0$ is a large arbitrary number. We will call them “asymptotic coordinates”. For such defined variables, neglecting the terms which is exponentially small with $|X|$ we have from Eq. (29):

$$\mu(x_{\leftarrow}, t) = -g(x_{\leftarrow}, t)^2, \quad \mu(x_{\rightarrow}, t) = g(x_{\rightarrow}, t)^2, \quad (36)$$

$$D(x_{\leftarrow}, t) = g(x_{\leftarrow}, t)^2/2, \quad D(x_{\rightarrow}, t) = g(x_{\rightarrow}, t)^2/2, \quad (37)$$

that is, the factor $\tanh(x)$ which were present in the diffusion coefficient in Eq. (32), disappears. In the following, we will consider only the case when $x \rightarrow -\infty$, and, correspondingly, we restrict ourselves to the variable x_{\leftarrow} (cf. Fig. 2). The dynamics for the case of $x \rightarrow +\infty$ is obviously analogous, only the overall drift direction will be the opposite as Eq. (36) indicates. The asymptotic FP equation for this case coincides with the original one Eqs. (27)-(28), only written in asymptotic coordinates $x \rightarrow x_{\leftarrow}$:

$$\partial_t P(t, x_{\leftarrow}) = -\partial_{x_{\leftarrow}} J(t, x_{\leftarrow}), \quad (38)$$

$$J(x_{\leftarrow}, y) = \mu(t, x_{\leftarrow}) P(t, x_{\leftarrow}) - \partial_{x_{\leftarrow}} (D(t, x_{\leftarrow}) P(t, x_{\leftarrow})). \quad (39)$$

C. FP equation for periodically varying potential

In this section we focus on the case when $g(x_{\leftarrow}, t)$ changes in space and time periodically. We assume g to have period L in space x_{\leftarrow} . In our new asymptotic coordinates, reformulation of the FP equation Eqs. (36)-(39) allowing to take advantage of such periodicity is possible [28]. Namely, we define the reduced quantities:

$$\tilde{P}(x_{\leftarrow}, t) = \sum_{n=-\infty}^{+\infty} P(x_{\leftarrow} + L, t), \quad (40)$$

$$\tilde{J}(x_{\leftarrow}, t) = \sum_{n=-\infty}^{+\infty} J(x_{\leftarrow} + L, t). \quad (41)$$

Obviously, $\tilde{P}(x_{\leftarrow}, t)$ and $\tilde{J}(x_{\leftarrow}, t)$ are finite and defined in the range $x_{\leftarrow} \in [-L/2, L/2]$. Moreover, from Eqs. (40)-(41) one can see that \tilde{P} , \tilde{J} are periodic in x_{\leftarrow} :

$$\tilde{P}(x_{\leftarrow}, t) = \tilde{P}(x_{\leftarrow} + L, t), \quad \tilde{J}(x_{\leftarrow}, t) = \tilde{J}(x_{\leftarrow} + L, t). \quad (42)$$

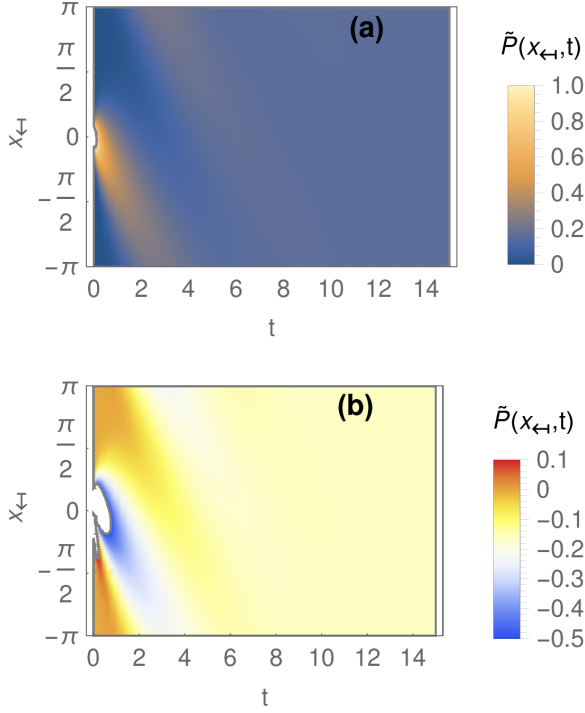


FIG. 3. The dynamics of the reduced asymptotic probability density $\tilde{P}(t, x_{\leftarrow})$ (a) and the current density $\tilde{J}(t, x_{\leftarrow})$ (b) for $g(t, x_{\leftarrow}) = 1$ obtained by direct simulations of Eqs. (43)-(44), assuming periodic boundary conditions and initial conditions described in text. In contrast to initial variables P, J , the reduced variable \tilde{P}, \tilde{J} do have a steady-state, which is in this case a homogeneous distribution.

In the asymptotic variables $\{x_{\leftarrow}, t\}$, as it follows from Eqs. (36)-(37), $\mu(x_{\leftarrow}, t)$ and $D(x_{\leftarrow}, t)$ are periodic in space with the same period L (which is by the way not true for μ and D written using the original variable x). Under these circumstances the FP equation written for \tilde{P}, \tilde{J} remains the same as for P, J . That is, we have:

$$\partial_t \tilde{P}(t, x_{\leftarrow}) = -\partial_{x_{\leftarrow}} \tilde{J}(t, x_{\leftarrow}), \quad (43)$$

$$\tilde{J}(x_{\leftarrow}, t) = \mu(t, x_{\leftarrow}) \tilde{P}(t, x_{\leftarrow}) - \partial_{x_{\leftarrow}} (D(t, x_{\leftarrow}) \tilde{P}(t, x_{\leftarrow})), \quad (44)$$

The advantage of such reformulation is that now we can consider only the finite interval in x_{\leftarrow} from, say, $-L/2$ to $L/2$. Besides, the equation for the average drift velocity Eq. (31) also retains its form:

$$\langle \dot{x}_{\leftarrow} \rangle = \int_{-L/2}^{L/2} \tilde{J}(x_{\leftarrow}, t) dx. \quad (45)$$

Remarkably, the direct definition of $\langle \dot{x}_{\leftarrow} \rangle$ as the average of \dot{x}_{\leftarrow} with the probability distribution $\tilde{P}(x_{\leftarrow}, t)$ is not valid anymore.

As an illustration of the dynamics appearing in the reduced equations, we show in Fig. 3 the dynamics of

$\tilde{P}(x_{\leftarrow}, t), \tilde{J}(x_{\leftarrow}, t)$ for the case of $g(x_{\leftarrow}, t) = \text{const} = 1$ obtained using direct numerical simulations of Eqs. (43)-(44) with the initial condition $P(x_{\leftarrow}, t) \propto \exp(-x_{\leftarrow}^2/0.1)$ and periodic boundary conditions. The figure shows rather rapid homogenization of $\tilde{P}(x_{\leftarrow}, t), \tilde{J}(x_{\leftarrow}, t)$ in space because of the action of diffusion. This homogenization illustrates an important peculiarity of the reduced quantities: although the initial variables J, P have no steady-state in their dynamics, the reduced quantities \tilde{J}, \tilde{P} do have a steady-state. In the case of Fig. 3 this steady state is simply a constant which does not depend neither on t nor on x_{\leftarrow} .

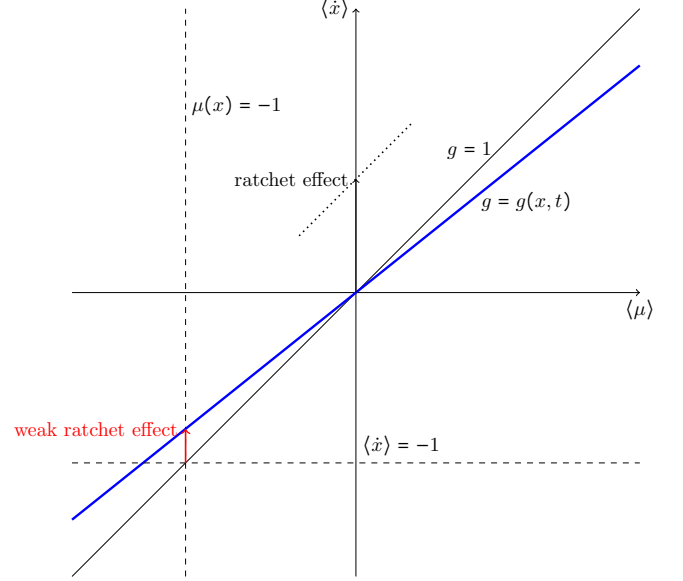


FIG. 4. Schematic representation of the Brownian ratchet effect. Without a ratchet effect ($g = 1$), a constant drift force μ leads to a current $\langle \dot{x} \rangle = \mu$ (black thin line). In contrast, when g changes in space and time (but still $\langle g \rangle = 1$), the average current $\langle \dot{x} \rangle$ can be modified even through the average force $\langle \mu \rangle$ remains the same (blue line). In many other systems the Brownian ratchet effect changes the average direction of motion (see dotted line), which is however not possible in here. Instead, the weak version of the ratchet effect is present, when $\langle \dot{x} \rangle$ is reduced but its direction is not reversed as represented by the blue line. The asymptotic values of μ and $\langle \dot{x} \rangle$ for $x \rightarrow -\infty$ (that is, assuming asymptotic coordinates x_{\leftarrow}) are marked by dashed lines. Weak ratchet effect in the asymptotic case is marked by a red arrow.

IV. BROWNIAN RATCHETS

One of the most interesting phenomena in Brownian flows is a possibility of so called stochastic ratchets [27, 28]. Namely, by manipulating dynamically the potential $V(x, t)$ in a Brownian flow, one can have nonzero average motion $\langle \dot{x} \rangle \neq 0$ even in the case when the average force $\langle \mu \rangle = \langle \int V(x) dx \rangle$ is exactly zero. Or, alternatively, if

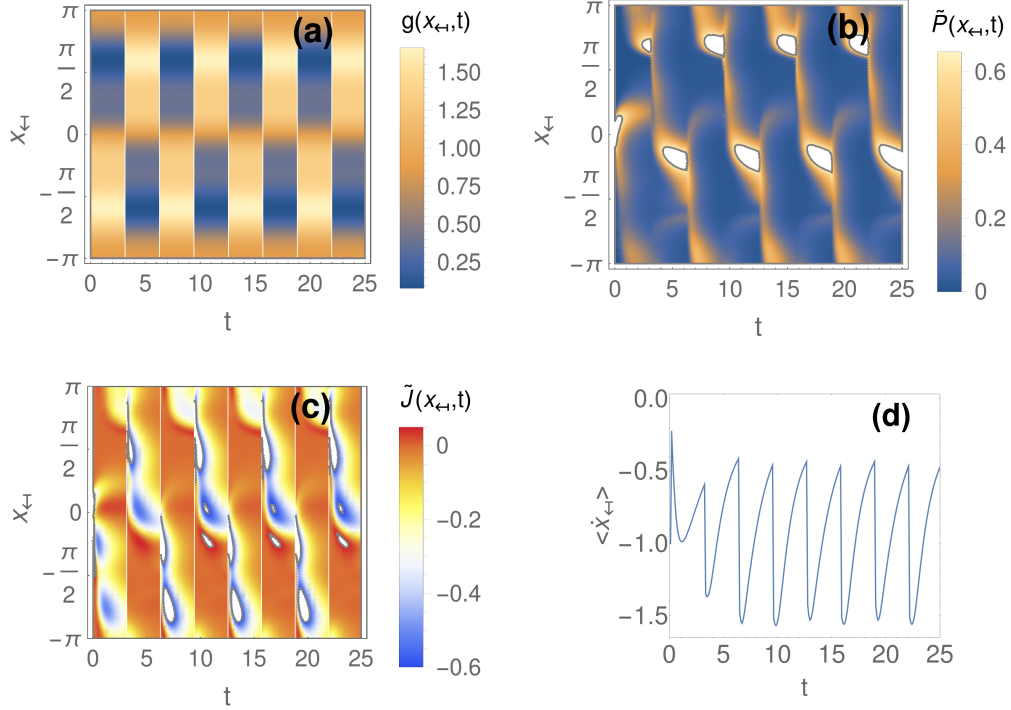


FIG. 5. Brownian ratchet effect for $g(x_{\leftarrow}, t)$ varying in space and time. (a) $g(x_{\leftarrow}, t)$, as given by Eq. (48). (b), (c) the reduced asymptotic probability density $\tilde{P}(x_{\leftarrow})$ and the current density $\tilde{J}(x_{\leftarrow})$ obtained by direct simulations of Eqs. (43)-(44) with periodic boundary conditions and initial conditions described in text. (d) the spatially averaged current $\langle \dot{x}_{\leftarrow} \rangle$ in dependence on time.

there is nonzero initial force $\mu \neq 0$, one can cancel it by introducing some modulations of the potential with $\langle \int V(x) dx \rangle = 0$.

In our case it is quite clear that the average flow defined by $\mu = -1$ (in the asymptotic case $x \rightarrow x_{\leftarrow}$) can not be reversed. Otherwise, one would have a possibility to violate the conservation law of $\langle \Pi \rangle$ given by Eq. (19) by tuning, for every particular trajectory, the potential in such a way that the current system state is forced to move in the direction opposite to μ and thus bring our system to any of the states $|0\rangle$, $|1\rangle$ at our wish which would violate Eq. (19).

Nevertheless, one can try to find a Brownian ratchet effect in a weak sense, that is, to find such a function $g(x_{\leftarrow}, t)$ that the asymptotic value of $\langle \dot{x}_{\leftarrow} \rangle > -1$, despite of $\mu = -1$ for $g = 1$. The notion of a weak ratchet effect, in comparison to a “normal” stochastic ratchet, is visualized in Fig. 4. Weak ratchets are in close correspondence to the weak Parrondo games, where the combination of lossy games lead to less lossy one, but still not to a winning one [30]. Of course, one can always obtain $\mu(x_{\leftarrow}, t) = 0$ by simply putting $g = 0$, that is, by reducing step size of the random walk to zero. Here we want however to investigate the effects which is independent on such raw step size reduction. To do this, we will assume that

$$\langle g(x_{\leftarrow}, t)^2 \rangle = 1, \quad (46)$$

where we define $\langle g(x_{\leftarrow}, t)^2 \rangle$ as:

$$\langle g(x_{\leftarrow}, t)^2 \rangle \equiv \frac{1}{L} \int_{-L/2}^{L/2} g(x_{\leftarrow}, t)^2 dx_{\leftarrow}. \quad (47)$$

This condition excludes the possibility to reduce $\langle \dot{x}_{\leftarrow} \rangle$ by reducing the measurement strength globally. That is, if one reduces the measurement strength near some point, one has to increase it in the vicinity of some another one.

We will now try to construct a periodic in time and space function g which allows to reduce $\langle \dot{x}_{\leftarrow} \rangle$, making it as small as possible. Namely, we will be interested in asymptotic value of $\langle \dot{x}_{\leftarrow} \rangle_{t \rightarrow +\infty}$. We remark that several various types of stochastic ratchets has been considered in the literature (see [28] and references therein), depending on how exactly D and μ are connected to each other. In contrast to our situation, in a typical hydrodynamic Brownian flow μ and D can be varied quite independently, thus giving rise, for instance, to a temperature ratchet, when $\mu = \text{const}$ and D is varied in space and optionally in time, or to pulsating ratchets, where μ changes and D is a constant. In our case, as seen from Eqs. (36)-(37), independent variation of D and μ is impossible because of the common factor g . Such situation somewhat resembles hydrodynamic Brownian ratchets with varying friction [28], which, in turn, can be reduced to so called tilting ratchets, that is, to the ratchets where μ is affected by an additive perturbation F : $\mu \rightarrow \mu + F(x, t)$.

Furthermore, one can see that if we will change g only in time (that is, assuming $g = g(t)$), no ratchet effect can appear. Namely, under this condition Eq. (45) can be calculated directly by integrating Eq. (44) with the boundary conditions Eq. (42), giving $\langle \dot{x}_{\leftarrow} \rangle = -1$. In this case $\tilde{P} \rightarrow \text{const} = 1/L$, that is, full homogenization of \tilde{P} will take place, exactly as in the case of $g = 1$.

Having in mind said above, we probe functions $g(x_{\leftarrow}, t)$ which varies both in time t and space x_{\leftarrow} . In Fig. 5 the dynamics for a deliberately taken $g(x_{\leftarrow}, t)$ of such type is shown, with g defined as (see Fig. 5(a)):

$$g = C(t) \{1 - 0.8 [\sin(x_{\leftarrow}) + 0.2 \sin(4x_{\leftarrow})] \text{sign}(\sin(t))\}, \quad (48)$$

where the normalizing constant $C(t)$ is obtained from Eq. (46). The boundary and initial conditions were taken as in Fig. 3.

One can see from Fig. 5(b) that, except of the initial short transient stage to $t \lesssim 3.5$, the probability density \tilde{P} concentrates in the regions where g (and thus D , μ) is minimal. This region is different at every cycle of g , which leads to the fast reallocation of \tilde{P} at every cycle. Part of the distribution moves, by this reallocation, to the lower values of x_{\leftarrow} whereas the another part moves in the direction of higher x_{\leftarrow} . This creates the current \tilde{J} which is mostly negative but sometimes is even slightly positive as seen from Fig. 5(c). In Fig. 5(d) one can see that the average current $\langle \dot{x}_{\leftarrow} \rangle$, after a short transition process, approaches to a stationary regime of oscillations in time with a period 2π . The long time behavior of the average of $\langle \dot{x}_{\leftarrow} \rangle$ over the time t is shown in Fig. 6, where it is seen that this average approaches ≈ -0.96 instead of -1 as in the case of constant $g = 1$, $\mu = -1$, thus clearly showing the ratchet effect in this process. As said, the condition Eq. (46) excludes the effect of bare step size reduction in this random walk, demonstrating that the ratchet effect is a dynamical phenomenon independent from the step size.

V. DYNAMICAL LOCALIZATION

Another interesting phenomenon is a dynamical localization of \tilde{P} in the presence of spatial variation of $g = g(x_{\leftarrow})$. To observe it, we assume that g depends only on x_{\leftarrow} but not on t . An exemplary profile of g which we use in this case is given by:

$$g(x_{\leftarrow}) = C(t)(1 - 0.8 \sin x_{\leftarrow}), \quad (49)$$

and shown in Fig. 7(a). Here the normalization constant $C(t)$ is obtained from Eq. (46) (and is different from $C(t)$ in Eq. (48)). For such function g , as one can see in Fig. 7(b), the average current $\langle \dot{x}_{\leftarrow} \rangle$ can be also larger than -1 ; in the case of Fig. 7 it approaches ≈ -0.2 as one can see in Fig. 7(d). In this case, the initial distribution is quickly rearranged to a stationary but inhomogeneous one.

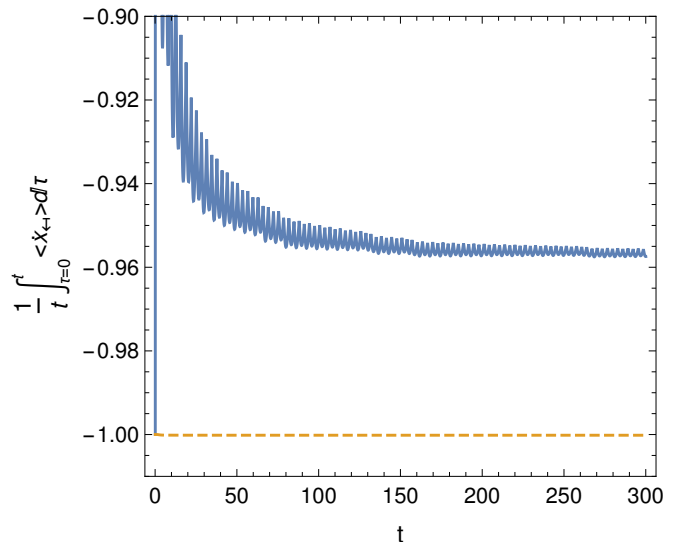


FIG. 6. Dependence of the temporal average of $\langle \dot{x}_{\leftarrow} \rangle$, that is, of $\frac{1}{t} \int_0^t \langle \dot{x}_{\leftarrow}(\tau) \rangle d\tau$ on the averaging time interval t . In the case of constant $g = 1$ (orange dashed line) this quantity quickly approaches -1 (no ratchet effect), whereas in the case of varying measurement strength Eq. (48) (blue solid line) it approaches ≈ -0.96 , demonstrating a weak Brownian ratchet.

The system now is located mostly near the minimum of g . The origin of such dynamics becomes more clear if one considers the potential $V(x_{\leftarrow})$ as shown in Fig. 8 (solid blue line). One can see that $V(x_{\leftarrow})$ approaches a flat region (where it is almost constant) close to $x_{\leftarrow} = \pi/2$. That is, there is almost no effective force at that point. If our effective “particle” approaches this region, it nearly stops. Nevertheless, in general the “particle” experiences some drift to the negative direction of x_{\leftarrow} .

VI. STRONG MEASUREMENT LIMIT

Finally, we consider the case when $g \rightarrow 0$ as x approaches some given X . That is, we take the limiting case of the potential considered in the previous example (where g approached small value which however was not zero). Besides, we now come back from “asymptotic coordinate” x_{\leftarrow} to the initial coordinate x and thus to FP equation as written in Eqs. (27)-(28). We assume also that $g(x)$ approaches zero only in one single point for simplicity.

As our previous example shows, we could expect the concentration of the probability density near $x = X$. Moreover, as both D and μ in X are zero, the trajectory can not cross X , and, as one can show, the time needed to achieve X is also infinite (see Appendix D for details). That is, as illustrated in Fig. 9(b), an ensemble of trajectories which are initially located at $x < X$ is divided into two separated parts: one consisting of the trajectories flying to $x \rightarrow -\infty$, whereas the other part

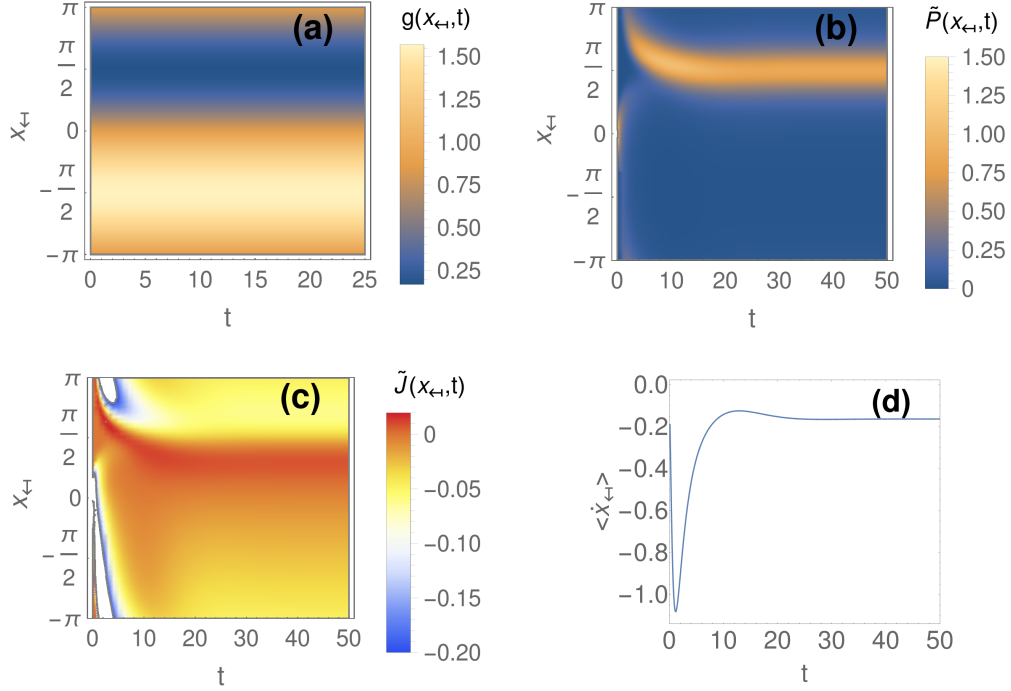


FIG. 7. Dynamical localization effect for $g(x_-)$ dependent only on spatial coordinate. (a) $g(x_-, t)$, as given by Eq. (49). (b), (c) the reduced asymptotic probability density $\tilde{P}(x_-)$ and the current density $\tilde{J}(x_-)$ obtained by direct simulations of Eqs. (43)-(44) with periodic boundary conditions and initial conditions described in text are shown. (d) the spatially averaged current $\langle \dot{x}_- \rangle$ in dependence on time.

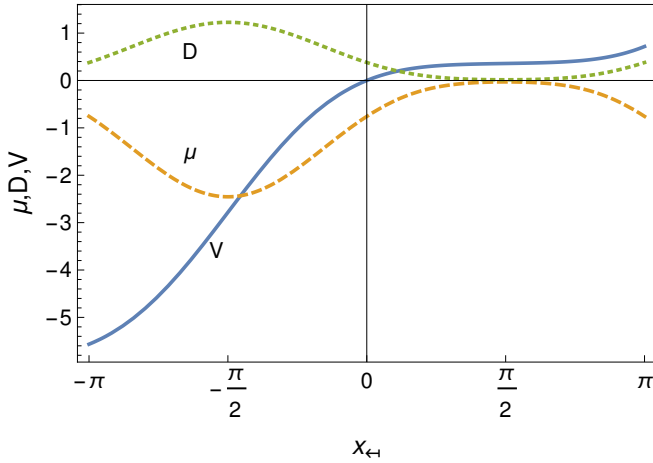


FIG. 8. Diffusion $D(x)$ (solid blue line), shift $\mu(x)$ (dashed red line) and the effective potential $V(x)$ (dotted yellow line, normalized to a constant $c = 0.1$ for better visibility) in dependence on x_- according to Eqs. (36)-(37) with $g(x_-)$ given in Eq. (49).

approach $x \rightarrow X$. That is, the state starting in between of $|0\rangle$ and $|X\rangle$ will approach either $|0\rangle$ or $|X\rangle$ as $t \rightarrow \infty$. Analogously, the state starting in between $|X\rangle$ and $|1\rangle$ will approach either $|X\rangle$ or $|1\rangle$. The state $|X\rangle$ is given

by

$$|X\rangle = \Pi_{X,0} |0\rangle + \Pi_{X,1} |1\rangle; \quad (50)$$

$$\Pi_{X,1} = \sqrt{\Pi(X)}, \quad \Pi_{X,0} = \sqrt{1 - \Pi(X)}, \quad (51)$$

where $\Pi(X)$ is given by Eq. (11).

As it is known, all von Neuman measurements, and, more generally, all generalized measurements described by positive operator-valued measure (POVM), can be produced as a limit $t \rightarrow \infty$ of the appropriately constructed weak measurement sequence with $g = \text{const}$ [17, 20, 21]. We now consider such limit for the weak measurement sequence we have just constructed, with $g \rightarrow 0$ at some $|X\rangle$.

How could a POVM look like, defined by such hypothetical measurement apparatus? Since $|X\rangle$ is a limiting state which can not be crossed, it becomes a kind of a new “basis state”. It, together with the “true” basis states $|0\rangle$ and $|1\rangle$ form now rather unusual POVM measure:

$$\mathcal{B}_0 = (1 - \Pi_{X,0}^2) |0\rangle \langle 0|, \quad (52)$$

$$\mathcal{B}_X = |X\rangle \langle X|, \quad (53)$$

$$\mathcal{B}_1 = (1 - \Pi_{X,1}^2) |1\rangle \langle 1|. \quad (54)$$

$$(55)$$

The operators \mathcal{B}_i , $i = 0, X, 1$, make the projections to the limiting states of the corresponding dynamics; Clearly,

they satisfy the usual condition $\sum_{i=0,1,X} \mathcal{B}_i \mathcal{B}_i^\dagger = 1$.

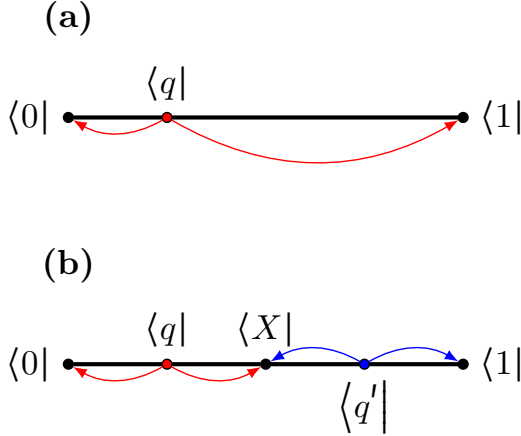


FIG. 9. Strong measurement limit of the weak measurement sequence in the case of $g = \text{const}$ (a) and in the case of $g(x, t)$ such that $g(x) \rightarrow 0$ as $x \rightarrow X$ (b); the coordinate X corresponds to the qubit state $|X\rangle$. In the former case, the measurement results in a collapse of an arbitrary state $|q\rangle$ either to $|0\rangle$ or $|1\rangle$, whereas in the case (b) the “collapse” to $|X\rangle$ also takes place. Some states (such as $|q\rangle$) collapse to either $|0\rangle$ and $|X\rangle$, the others (as $|q'\rangle$) collapse to $|1\rangle$ and $|X\rangle$.

VII. CONCLUSIONS

In the present article we have considered quantum trajectories resulting from a sequence of weak measurements, in the simplest one-dimensional settings, but assuming the measurement strength depending on the step number n and on the current state of the system described by the coordinate x on the line.

Such measurement process, in the limit of infinitely small steps, leads to a diffusive dynamics with the both drift and diffusion depending on the coordinate x and time t . In fact, the dynamics arising in such case is quite similar to, for instance, overdamped Brownian particle in a flow with varying friction coefficient. In this article we discussed the nontrivial dynamics arising due to this analogy.

For instance, an exciting phenomenon arising in Brownian flows is the stochastic ratchet effect, which allows to “rectify” Brownian motion using periodically varying potential. Such potential does not introduce any net force by itself, nevertheless allowing to push particles in the direction opposite to the flow. In our case, we can achieve only a weak form of the stochastic ratchet effect. That is, we can not reverse the overall drift direction of the quantum trajectories, but only slow down this motion. We showed also a possibility of the state localization in the areas where the measurement strength is reduced.

Finally, we considered the case when the step size approaches zero as the system approaches some $|X\rangle$. No quantum trajectory can cross the singularity point arising in this case. Moreover, the trajectories approach such singularity in infinite time in a similar way as they approach the “normal” basis states. This allows to introduce a general measurement process as the limiting case of such unusual weak measurement sequence.

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Appendix A: Derivation of the expression for step size

To derive the step size $\epsilon_i(x_n) \equiv x_{n+1} - x_n$ on the n th step of our random walk, we use Eqs. (3)-(7) and the relation:

$$\sin^2 \theta_n = \frac{1 + \tanh x_n}{2}. \quad (\text{A1})$$

For instance, in the case if the measurement of the ancilla $|a\rangle$ is $|0\rangle$, we have from Eqs. (3)-(7):

$$\sin \theta_{n+1} = \sin \theta_n \cos(\delta + \alpha) / \sqrt{p_0}, \quad (\text{A2})$$

and thus, using Eq. (A1):

$$\frac{1 + \tanh x_{n+1}}{2} = \frac{(1 + \tanh x_n) \cos^2(\delta + \alpha)}{2p_0}. \quad (\text{A3})$$

Hence, the expression for $\epsilon_0(x)$ (redefining x_n as x since x_n is arbitrary) is:

$$\epsilon_0(x) = \text{atanh} \left(\frac{(1 + \tanh x) \cos^2(\delta + \alpha)}{p_0(x)} - 1 \right) - x. \quad (\text{A4})$$

In the same way, if $|a\rangle$ collapses to $|1\rangle$ upon the measurement, we have

$$\sin \theta_{n+1} = \sin \theta_n \sin(\delta + \alpha) / \sqrt{p_1}, \quad (\text{A5})$$

and thus we obtain for $\epsilon_1(x)$:

$$\epsilon_1(x) = \text{atanh} \left(\frac{(1 + \tanh x) \sin^2(\delta + \alpha)}{p_1(x)} - 1 \right) - x. \quad (\text{A6})$$

Now we calculate analytically the expression for $d\epsilon_i(x)/dx$, which can be straightforwardly shown to be zero. Thus, $\epsilon_i(x) = \epsilon_i(0) \equiv \epsilon_i$ and we can take $x = 0$ in Eq. (A4), Eq. (A6), thus obtaining expressions Eqs. (14)-(15).

Appendix B: Quantum trajectories with constant measurement parameters

In the case of a constant x -independent step (and only in this case) it is constructive to analyze ϵ_i , p_i on more general level by introducing the “average step” $\mu(x)$:

$$\mu(x) = \mu_0(x) + \mu_1(x), \mu_i(x) = \epsilon_i p_i(x), i = 0, 1. \quad (\text{B1})$$

μ defines an “average direction” of evolution: towards $+\infty$ or $-\infty$.

Equations Eq. (B1) are simplified when δ is small (assuming fixed $\alpha > 0$), so we can decompose Eqs. (12)-(15) in series in δ . In this case, up to the second order of δ we have:

$$\mu(x) = \delta^2 \tanh(x) + O(\delta^3), \quad (\text{B2})$$

$$2\mu_0(x) = \delta \sin(2\alpha) + \delta^2 (4\Pi(x) \sin^2 \alpha - 1) + O(\delta^3), \quad (\text{B3})$$

$$2\mu_1(x) = -\delta \sin(2\alpha) + \delta^2 (4\Pi(x) \cos^2 \alpha - 1) + O(\delta^3), \quad (\text{B4})$$

where $\Pi(x)$ is given by Eq. (11). Since $\text{sign } x = \text{sign } \tanh x$ Eq. (B2) demonstrates a “weak attraction” of the dynamics to the nearest state. We also can define in this case a quantity D , which have the meaning of a diffusion coefficient:

$$D(x) = \frac{1}{2} \sum_i p_i(x) \epsilon_i^2. \quad (\text{B5})$$

It is easy to see that in the limit of small δ (assuming $\alpha = \text{const}$) we have:

$$D(x) = \frac{1}{2} \delta^2 + O(\delta^3). \quad (\text{B6})$$

Appendix C: Derivation of the Fokker-Planck equation

We derive the FP equation using the standard integral approach [40]. Namely, we consider an arbitrary function $h(x)$ which has a finite support, that is, localized inside the integration area and is zero together with all of its derivatives for large enough $|x|$. We also assume that it is smooth enough. Then, we write the expression for $\int h(x) \partial_t P(t, x) dx$, assuming integration over the whole real axis:

$$\tau_n \int h(x) \partial_t P(t, x) dx \cong \int h(x) (P(t + \tau, x) - P(t, x)) dx, \quad (\text{C1})$$

Expressing $P(t + \tau_n, x)$ through $P(t, x)$ using Eq. (17) and assuming $t = \sum_n \tau_n$, replacing variables in two integral parts as $x \rightarrow x_i(x)$ followed by redefining $x_i \rightarrow x$, and finally expanding $h(x + \epsilon) \cong h(x) + \epsilon h'(x) + \epsilon^2 h''(x)/2$, we transform the later expression into:

$$\int P(t, x) \left(\sum_i p_i (h'(x) \epsilon_i(x) + h''(x) \epsilon_i^2(x)/2) \right) dx. \quad (\text{C2})$$

Applying integration by parts we have finally:

$$\int h(x) \{ -\partial_t P(t, x) - \partial_x [\mu(x, t) P(t, x)] + \partial_{xx} [D(x, t) P(t, x)] \} dx = 0, \quad (\text{C3})$$

where

$$\mu(x, t) = \sum_i \frac{p_i(x, n) \epsilon_i(x, n)}{\tau_n} \Big|_{n \rightarrow t}, \quad (\text{C4})$$

$$D(x, t) = \sum_i \frac{p_i(x, n) \epsilon_i^2(x, n)}{2\tau_n} \Big|_{n \rightarrow t}. \quad (\text{C5})$$

For small δ and α_n being constant for every n we have, up to the second order of δ :

$$\sum_i p_i(x, t) \epsilon_i(x, t) = \delta_n(x)^2 \tanh(x) + O(\delta_n(x)^3), \quad (\text{C6})$$

$$\sum_i p_i(x, t) \epsilon_i^2(x, t) = \delta_n(x)^2 + O(\delta_n(x)^3). \quad (\text{C7})$$

Taking into account Eq. (24), we finally arrive to Eqs. (26)-(29).

Appendix D: Asymptotic rate of approaching the “artificial” pure state

Suppose that $g(x)$ is decomposed near X as: $g = g_0(x - X) + \dots$, with $g_0 \neq 0$. In the following we also assume $X \neq 0$. In this case, in the vicinity of X we have from Eq. (29): $\mu = \mu_0(x - X)^2 + \dots$, $D = D_0(x - X)^2 + \dots$, for some μ_0, D_0 . To test how x approaches X as $t \rightarrow \infty$, one could make an expansion of the vicinity of the singular point, analogous to the one we made when going from θ to x coordinates. That is, we replace the vicinity of a singular point by a (semi-)infinite interval. To do this analytically, we again assume for simplicity the asymptotic coordinates x_{\leftarrow} introduced in Eq. (34), and take g as a linear function $g = g_0 x_{\leftarrow}$, $g_0 > 0$. In this case, the singularity ($\mu = 0$, $D = 0$) appears at $x_{\leftarrow} = 0$. We also assume that $x_{\leftarrow} > 0$, that is, we are approaching singularity from the negative direction. We will now find the new coordinates $y(x_{\leftarrow})$ in which $D(y) = 1/2$ and thus make the case comparable to a standard diffusion process. Taking into account the variable replacement rules in FP equation:

$$\mu(y) = \mu(x_{\leftarrow}(y)) y' + D(x_{\leftarrow}(y)) y'', \quad (\text{D1})$$

$$D(y) = D(x_{\leftarrow}(y)) (y')^2, \quad (\text{D2})$$

(where by y' , y'' the derivatives in respect to x_{\leftarrow}), the relation $D(x_{\leftarrow}(y)) = g_0^2 x_{\leftarrow}^2 / 2$ and the fact that we want to have in the new coordinates $D(y) = 1/2$, we have $y' = g_0 / x_{\leftarrow}$ and thus:

$$y(x_{\leftarrow}) = \ln x_{\leftarrow} / g_0. \quad (\text{D3})$$

That is, $y \rightarrow -\infty$ as $x_{\leftarrow} \rightarrow 0$. With this, we have:

$$\mu(y) = -g_0/2 - e^{g_0 y}. \quad (\text{D4})$$

The last term in Eq. (D4) quickly disappears as we approach the singularity, thus giving us for $y \ll -1$ $\mu(y) \approx -g_0/2$. Finally, the new FP equation with coefficients Eqs. (D1)-(D2) can be re-scaled as $y \rightarrow 2y/g_0$, $t \rightarrow g_0^2/4$ to give us:

$$\mu(y) \approx -1, D(y) \approx 1/2, \quad (\text{D5})$$

as $y \rightarrow -\infty$. That is, in the y -coordinates we have the quite standard FP process with the asymptotically constant drift and diffusion. The singularity point $y = -\infty$ will be thus achieved clearly in the infinite time. This asymptotic behavior is common for any function g which approaches to zero as $g = g_0(x - X) + g_1(x - X)^2 + \dots$ with $g_0 \neq 0$.

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